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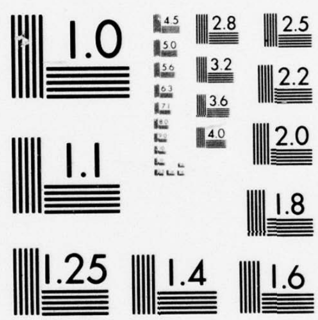

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STRONG CONVERGENCE OF SEMIGROUPS OF NONLINEAR CONTRACTIONS IN HILBERT SPACE

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ABSTRACT

The present paper deals with the strong convergence of trajectories $S(t)x$ of a strongly continuous semigroup of contractions $S(t)$, as $t \to \infty$. A general sufficient condition for such convergence to occur is introduced and some examples in which the condition is satisfied are provided. Strengthening the general convergence condition, sufficient conditions for certain rates of convergence of $S(t)x$ to its limit are exhibited. In particular a sufficient condition for a trajectory to reach equilibrium in finite time is given. The convergence as $t \to \infty$ of solutions of certain nonautonomous equations and a discrete version of all the previous results are briefly discussed.

AMS (MOS) Subject Classifications: 47H15, 35B40
Key Words: Semigroups of contractions, Fixed points, Subdifferential
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See 1473

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SIGNIFICANCE AND EXPLANATION

Many of the phenomena in nature are governed by a special type of equation called an evolution equation. What characterizes these equations is that the state of the system, described by the equation, at the time t uniquely determines the whole future evolution of the system. Some examples of such evolution processes are: The motion of a pendulum, temperature distribution in a conducting body, diffusion of salt in water, certain flows of fluids etc. The equations describing each one of these phenomena are of course different but they all have in common the property mentioned above namely, the state of the system at each time t determines the whole future uniquely.

One of the natural questions that arises concerning such evolution systems is what happens to the system after a long time (or as time tends to infinity). Since the whole future is determined by the equation and the initial conditions which are known to us, we should be able to predict the behavior of the system as time goes to infinity, or in other words the asymptotic behavior of the system.

An evolution system may have different types of behavior as $t \to \infty$. One of the most common behaviors is that the solution converges to a stationary (i.e. time independent) solution of the problem. For example the temperature in an insulated body without sources of heat will tend exponentially to a constant temperature. The motion of a pendulum, taking into account friction, will eventually stop. If the friction is small it will be only as $t \to \infty$ that the pendulum will stop. If the friction is "large" it will stop in finite time.

In this paper we deal with the asymptotic behavior of a class of evolution equations. We give conditions on the equations that guarantee that the solution will tend as $t \to \infty$ to a stationary solution, whatever the initial data are. We study the rate of convergence to this stationary state. In particular we give conditions for which the stationary state is attained in finite time.

As we mentioned above, the initial data determine the whole future of the system uniquely. Therefore in principle it should be possible to determine a-priori the limiting state of the system for any given initial conditions. In certain simple cases this is indeed the case. In general however, it is very difficult to predict the limit without solving the equations. It is even difficult to give nontrivial estimates on the "location" of this limiting solution. We discuss this problem in the present paper via a specific example of heat conduction.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STRONG CONVERGENCE OF SEMIGROUPS OF NONLINEAR CONTRACTIONS IN HILBERT SPACE

A. Pazy[†]

§1. Introduction

Let H be a real Hilbert space, C a closed convex subset of H and $S(t) : C \to C$ a strongly continuous semigroup of contractions on C. The purpose of the present paper is to study the strong convergence of the trajectories $S(t)x$, $x \in C$ of $S(t)$ as $t \to \infty$.

In Section 2 we introduce a rather general condition on the generator A of a semigroup $S(t)$, called the convergence condition. This condition assures the strong convergence of $S(t)x$ as $t \to \infty$ for every $x \in C$. It can be seen by simple examples (e.g. example 4.5 of the present paper) that the convergence condition is not necessary in order to have strong convergence of $S(t)x$ as $t \to \infty$, for all $x \in C$. The convergence condition, introduced in Section 2, contains as special cases most of the previously known sufficient conditions for strong convergence of all trajectories of a semigroup of contractions in Hilbert space. The only known notable exceptions are semigroups generated by subdifferentials of l.s.c. (lower semicontinuous) even convex functions (see [6]) and semigroups having a fixed point set with nonempty interior (see [3], [8]).

In Section 3 we study the convergence condition more closely. We show that certain natural compactness assumptions on the resolvent of A together with a simple geometric condition on the tangent of the trajectory at each point imply the convergence condition. In particular we prove that the convergence condition is satisfied if A is the subdifferential of a l.s.c. convex function φ whose level sets are compact and as a consequence we obtain a result of H. Brezis [3]. The second part of Section 3 is devoted to some simple examples of the way that the abstract results of Section 2 can be applied to the study of the asymptotic behavior of certain nonlinear parabolic partial differential equations.

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In general, the convergence condition introduced in Section 2 does not imply any special rate of convergence. It is not difficult to construct examples in which the convergence is as slow as one wishes, or very fast. In many cases however, a slightly stronger condition, called the uniform convergence condition is satisfied. If this is the case, one can usually obtain some information on the rate of convergence of $S(t)x$ to its limit p as $t \to \infty$. The uniform convergence condition is the subject of Section 4. Among other consequences of the uniform convergence condition we give in Proposition 4.8, sufficient conditions for $S(t)x$ to attain its limit in finite time.

Section 5 is devoted to a brief consideration of the asymptotic behavior of solutions of certain nonautonomous initial value problems. In particular we show that if A satisfies the convergence condition, not only does every trajectory of $S(t)$, the semi-group generated by A, converge as $t \to \infty$ but also all solutions of the initial value problem

$$(1.1) \qquad \begin{cases} u_t + Au \ni f(t) \\ u(0) = x \end{cases}$$

with $f(t) \in L^1((0,\infty):H)$ converge strongly as $t \to \infty$.

In Section 6 we study a discrete version of the main convergence results of this paper. The results of this section are related to a recent paper of H. Brezis and P. L. Lions [5].

The convergence condition assures that $S(t)x \to p$ as $t \to \infty$ where p is a fixed point of $S(t)$. If the set F of fixed points of $S(t)$ contains more than one point, the natural question of the identification of the limit point p in terms of the initial value x arises. Very little is known in general on this difficult problem. In Section 7 we study in somewhat greater detail an example of a nonlinear Neumann problem. In this example we use the techniques developed in this paper together with some standard tools as the maximum principle to prove the convergence of the solutions of this problem to fixed points, to estimate the rate of this convergence and to obtain some a-priori estimates on the limit in terms of the initial data.

Finally, I would like to express my gratitude to H. Brezis and M. Crandall for several stimulating discussions concerning the results of this paper.

§2. The convergence theorem

Let H be a real Hilbert space with inner product $(,)$ and norm $| \ |$. Let A be a maximal monotone set in $H \times H$ and let $S(t)$ be the semigroup of contractions generated by A. For a systematic exposition of the theory of monotone operators and semigroups of contractions in Hilbert space see [2]. We denote by F the (possibly empty) set of fixed points of the semigroup $S(t)$, that is

$$F = \{x : x \in \overline{D(A)},\ S(t)x = x \quad \forall t \geq 0\} . \tag{2.1}$$

It is not difficult to see that

$$F = A^{-1}0 = \{x : x \in D(A),\ A^0x = 0\} \tag{2.2}$$

where A^0x is as usual the element of minimum norm in the set Ax. Since A^{-1} is maximal monotone together with A, F is always a closed and convex subset of $\overline{D(A)}$. If $F \neq \phi$ we shall denote by P the projection on F.

Definition 2.1:

A maximal monotone set $A \subset H \times H$ satisfies the convergence condition if:

a) $F = A^{-1}0$ is not empty.

b) $(y, x - Px) > 0$ for every $[x,y] \in A$ such that $x \notin F$.

c) If $[x_n, y_n] \in A$, $|x_n| \leq C$, $|y_n| \leq C$ and $(y_n, x_n - Px_n) \to 0$ as $n \to \infty$ then

$$\liminf_{n\to\infty} \operatorname{dist}(x_n, F) = 0$$

where $\operatorname{dist}(y,F)$ is the distance between the point y and the closed convex set F.

Remark: For reasons of later convenience we stated part (b) of the definition explicitly even though it is clearly implied by part (c).

Our main result is the following theorem.

Theorem 2.2:

Let A be maximal monotone and let $S(t)$ be the semigroup generated by A. If A satisfies the convergence condition then for every $x \in \overline{D(A)}$, $S(t)x$ converges strongly as $t \to \infty$ to a fixed point of $S(t)$.

In the proof of Theorem 2.2 we shall need the following two lemmas.

Lemma 2.3 (Baillon-Brezis [1]):

Let $S(t)$ be a semigroup of contractions on a closed convex subset $C \subset H$. If $F \neq \phi$ and P is the projection on F, then for every $x \in C$, $PS(t)x$ converges strongly as $t \to \infty$.

Proof: Since P is the projection on F, we have;

$$|Pv - u|^2 \leq |v - u|^2 - |Pv - v|^2 \quad \forall v \in H,\ u \in F \tag{2.3}$$

Substituting $v = S(t + h)x$ and $u = PS(t)x$ into (2.3) we obtain

$$|PS(t + h)x - PS(t)x|^2 \leq |S(t + h)x - PS(t)x|^2 - |S(t + h)x - PS(t + h)x|^2 \leq$$
$$\leq |S(t)x - PS(t)x|^2 - |S(t + h)x - PS(t + h)x|^2 .$$

Therefore, $t \mapsto |S(t)x - PS(t)x|^2$ is monotone nonincreasing and $PS(t)x$ is a Cauchy net. ∎

Lemma 2.4:

Let A be maximal monotone with $F = A^{-1}0 \neq \phi$. If $x \in D(A)$ then $PS(t)x$ is differentiable a.e. in t and

$$\left(\frac{dPS(t)x}{dt}, S(t)x - PS(t)x\right) = 0 \text{ a.e. in } t \geq 0 . \tag{2.4}$$

Proof: For $x \in D(A)$, $S(t)x$ is Lipschitz in t and since P is nonexpansive $PS(t)x$ is Lipschitz in t and therefore differentiable almost everywhere.

From the definition of P it follows that

$$(PS(t + h)x - PS(t)x,\ S(t)x - PS(t)x) \leq 0 . \tag{2.5}$$

Dividing (2.5) by $h > 0$ and $h < 0$ and letting $h \to 0$ the result follows. ∎

Proof of Theorem 2.2:

If $x \in D(A)$ then

$$\frac{dS(t)x}{dt} + A^0 S(t)x = 0 . \tag{2.6}$$

Multiplying (2.6) by $S(t)x - PS(t)x$ and using Lemma 2.4 we obtain

$$\frac{1}{2}\frac{d}{dt}|S(t)x - PS(t)x|^2 + (A^0S(t)x, S(t)x - PS(t)x) = 0 . \tag{2.7}$$

Since $(A^0S(t)x, S(t)x - PS(t)x) \geq 0$ it follows that $t \mapsto |S(t)x - PS(t)x|$ is monotone nonincreasing. From (2.7) it also follows that $(A^0S(t)x, S(t)x - PS(t)x) \in L^1(0,\infty)$.

Therefore there is a sequence $t_k \to \infty$ such that $(A^0 S(t_k)x, S(t_k)x - PS(t_k)x) \to 0$ as $t_k \to \infty$. Since $F \neq \phi$ implies that $|S(t)x|$ is bounded and $|A^0 S(t)x| \leq |A^0 x|$ for all $t \geq 0$ it follows from the convergence condition that $\liminf_{k\to\infty} |S(t_k)x - PS(t_k)x| = 0$ and since $t \mapsto |S(t)x - PS(t)x|$ is nonincreasing, $|S(t)x - PS(t)x| \to 0$ as $t \to \infty$. Finally, it follows from Lemma 2.3 that $|PS(t)x - p| \to 0$ as $t \to \infty$ for some $p \in F$ and therefore $S(t)x \to p$ as $t \to \infty$. This concludes the proof for $x \in D(A)$. For $x \in \overline{D(A)}$ the result follows from a simple continuity argument. ■

We turn now to a perturbation theorem. In order to state it we shall need the following definition.

Definition 2.5:

Two multi valued operators A and B are relatively locally bounded if for every $R > 0$ the boundedness of the set $\{(A + B)x : x \in D(A) \cap D(B), |x| \leq R\}$ implies the boundedness of the sets $\{Ax : x \in D(A) \cap D(B), |x| \leq R\}$ and $\{Bx : x \in D(A) \cap D(B), |x| \leq R\}$.

Note that if one of the two operators A or B is locally bounded i.e. it maps bounded sets into bounded sets then A and B are always relatively locally bounded.

Theorem 2.6:

Let A be maximal monotone and let B be monotone such that $A + B$ is maximal monotone. If A satisfies the convergence condition and

i) $B^{-1}0 = F_B \supset F_A = A^{-1}0$

ii) A and B are relatively locally bounded

then $A + B$ satisfies the convergence condition.

Proof: From our assumptions it follows that $F_{A+B} \supset F_A \cap F_B = F_A$ and therefore $F_{A+B} \neq \phi$. If $x \in F_{A+B}$ then there are $\eta_1 \in Ax$ and $\eta_2 \in Bx$ such that $\eta_1 + \eta_2 = 0$. Multiplying this equality by $x - P_A x$ where P_A is the projection on F_A, we have

$$(\eta_1, x - P_A x) + (\eta_2, x - P_A x) = 0 .$$

Since $P_A x \in F_B$, both terms must vanish and therefore by the convergence condition $x = P_A x$ i.e. $x \in F_A$. Therefore $F_{A+B} = F_A$ and $P_{A+B} = P_A$. Assume now that $[x_n, y_n] \in A + B$, $|x_n| \leq C$, $|y_n| \leq C$ and $(y_n, x_n - P_{A+B} x_n) \to 0$ as $n \to \infty$. Set

$y_n = \eta_n + \eta_n'$ where $\eta_n \in Ax_n$ and $\eta_n' \in Bx_n$ then

$$(\eta_n, x_n - P_A x_n) + (\eta_n', x_n - P_A x_n) \to 0$$

as $n \to \infty$. Since both terms are nonnegative we have $(\eta_n, x_n - P_A x_n) \to 0$. From (ii) we deduce that $|\eta_n| \leq R$ and therefore it follows from the convergence condition that there is a subsequence $\{n_k\}$ such that $|x_{n_k} - P_A x_{n_k}| \to 0$ and thus $A + B$ satisfies the convergence condition. ■

Remark 2.7:

In the previous theorem the condition that B is monotone can be replaced by the following assumption; there is a $k > 1$ such that $A + kB$ is monotone. To see this, note that in this case $k^{-1}A + B$ is monotone and defining $A_1 = (1 - k^{-1})A$, $B_1 = k^{-1}A + B$, B_1 is monotone and the pair A_1, B_1 satisfies the conditions of Theorem 2.6 therefore $A_1 + B_1 = A + B$ satisfies the convergence condition.

We conclude this section with a proposition showing that if A satisfies the convergence condition so does its Yosida approximation $A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1})$.

Proposition 2.8:

Let A be maximal monotone. If A satisfies the convergence condition, then for every $\lambda > 0$, A_λ satisfies the convergence condition.

Proof: It is easy to see that $F_{A_\lambda} = F_A$ and therefore $F_{A_\lambda} \neq \phi$ and the projection on F_{A_λ} is the same as the projection on F_A. We denote this projection by P. Let $|x_n| \leq C$, $|A_\lambda x_n| \leq C$ and,

(2.8) $$(A_\lambda x_n, x_n - Px_n) = (A_\lambda x_n, J_\lambda x_n - Px_n) + \lambda |A_\lambda x_n|^2 \to 0 \text{ as } n \to \infty .$$

Since $|A_\lambda x_n| \leq C$ also $|J_\lambda x_n|$ are bounded. From (2.8) we deduce that $|A_\lambda x_n| \to 0$ as $n \to \infty$ and that $(A_\lambda x_n, J_\lambda x_n - Px_n) \to 0$. Since $A_\lambda x_n \in AJ_\lambda x_n$ it follows from the convergence condition that $\liminf_{n\to\infty} |J_\lambda x_n - Px_n| = 0$ but $|x_n - J_\lambda x_n| = \lambda |A_\lambda x_n| \to 0$ as $n \to \infty$ and therefore $\liminf_{n\to\infty} |x_n - Px_n| = 0$ and the proof is complete. ■

§3. The convergence condition and some examples

In this section we shall see some sufficient conditions for a maximal monotone operator A to satisfy the convergence condition. After this we shall give some simple concrete examples for which the condition is satisfied.

Let φ be a proper convex lower semicontinuous (l.s.c.) function and let $A = \partial\varphi$ be its subdifferential. It is well known (see e.g. [2] example 2.3.4) that A is maximal monotone and we have,

Proposition 3.1:

Let φ be a proper convex l.s.c. function on H satisfying $\varphi(x) \geq 0$ and $\operatorname{Min}_{x \in H} \varphi(x) = 0$. If for every $R \geq 0$ the level sets

$$K_R = \{x : |x| \leq R, \varphi(x) \leq R\} \tag{3.1}$$

are precompact then $A = \partial\varphi$ satisfies the convergence condition.

Proof: Since by assumption, the minimum of $\varphi(x)$ is attained $F = A^{-1}0 \neq \phi$. From the definition of $\partial\varphi$ we have

$$\varphi(Px) - \varphi(x) = -\varphi(x) \geq (y, Px - x) \quad \forall [x,y] \in A$$

and therefore $(y, x - Px) \geq \varphi(x)$. Let $|x_n| \leq C$, $|y_n| \leq C$, $y_n \in Ax_n$ and $(y_n, x_n - Px_n) \to 0$. This implies $\varphi(x_n) \to 0$ and therefore $\varphi(x_n) \leq C$ for n large enough. Since $\{x_n\}$ lies in a precompact set K_C it has a converging subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x$, by the lower semicontinuity of φ it follows that $\varphi(x) = 0$ and therefore $x \in F$ and $\liminf_{n\to\infty} \operatorname{dist}(x_n, F) = 0$. ∎

Remark: The consequences of Proposition 3.1 namely, the strong convergence of $S(t)x$ as $t \to \infty$ under the assumptions of Proposition 3.1 were proved by H. Brezis ([2], theorem 3.11) by a different method.

Our next proposition is a generalization of the previous proposition to the case where A is no longer a subdifferential of a convex function.

Proposition 3.2:

Let A be a maximal monotone operator with $F = A^{-1}0 \neq \phi$. If for every $[x,y] \in A$, $x \notin F$, $(y, x - Px) > 0$ and $(I + A)^{-1}$ is a compact operator then A satisfies the convergence condition.

Proof: We have only to show that A satisfies part (c) of the convergence condition. We first check that if $(I + A)^{-1}$ is compact then for every $R \geq 0$ the set

$$E_R = \{x : x \in D(A),\ |x| \leq R,\ |y| \leq R \text{ for some } y \in Ax\} \tag{3.2}$$

is precompact. This follows immediately from the following observation; if M_R is the image of $B_R = \{x : |x| \leq R\}$ by the mapping $(I + A)^{-1}$ then $E_{R/2} \subset M_R$. Indeed $x \in E_{R/2}$ implies $|x| < R/2$, $|A^0 x| < R/2$ so if $z = x + A^0 x$, $|z| \leq R$ and $x = (I + A)^{-1} z$. Therefore $x \in M_R$.

Assume now that $[x_n, y_n] \in A$, $|x_n| \leq C$, $|y_n| \leq C$ and $(y_n, x_n - Px_n) \to 0$. From the compactness of E_C it follows that there is a subsequence x_{n_k} such that $x_{n_k} \to x$ as $n_k \to \infty$. Therefore, $Px_{n_k} \to Px$. Passing to a subsequence of $\{n_k\}$ if necessary, we can assume that y_{n_k} converges weakly to some y and deduce from the maximality of A that $[x,y] \in A$ and $(y, x - Px) = 0$. From our hypothesis it now follows that $x = Px$ i.e. $x \in F$ and so $\text{dist}(x_{n_k}, F) \to 0$ as $n_k \to \infty$. Therefore part (c) of the convergence condition is satisfied and the proof is complete. ∎

We turn now to some *simple examples*. *In these examples* Ω will be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. H will be the Hilbert space $L^2(\Omega)$, its norm will be denoted by $\|\cdot\|$ and β will be a maximal monotone graph with primitive $j(x)$, i.e. $j(x)$ is a proper convex lower semicontinuous function on $\mathbb{R}$ such that $\beta = \partial j$.

Example 3.3:

Let $0 \in \beta(0)$. In $H = L^2(\Omega)$ consider the operator A_0 defined by:

$$D(A_0) = \{u : u \in H^2(\Omega),\ \frac{\partial u}{\partial n} \in -\beta(u) \text{ a.e. on } \partial\Omega\} \tag{3.4}$$

and

$$A_0 u = -\Delta u \text{ for } u \in D(A_0)\ . \tag{3.5}$$

Here $H^2(\Omega)$ is the usual Sobolev space *consisting of all functions* u *which are in* $L^2(\Omega)$ together with all their second order distributional derivatives and n is the outward normal to $\partial\Omega$. It was shown in [3] that $A_0 = \partial\varphi_0$, where φ_0 is a proper convex l.s.c. function given by

$$(3.6)\qquad \varphi_0(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} j(u)d\sigma & \text{for } u \in H^1(\Omega) \text{ and } j(u) \in L^1(\partial\Omega) \\ +\infty & \text{otherwise .} \end{cases}$$

Therefore, in particular, A_0 is maximal monotone. From Rellich's compactness theorem it readily follows that the sets

$$(3.7)\qquad K_C = \{u : u \in L^2(\Omega),\ \|u\| \le C,\ \varphi_0(u) \le C\}$$

are precompact in $L^2(\Omega)$ for every real C. Also clearly $0 \in F_{A_0}$, therefore $F_{A_0} \neq \phi$ and we can apply Proposition 3.1 to show that for every $u_0 \in L^2(\Omega)$ the solution of the initial value problem

$$(3.8)\qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega \times (0,\infty) \\ -\frac{\partial u}{\partial n} \in \beta(u) & \text{on } \partial\Omega \times (0,\infty) \\ u(0,x) = u_0(x) \end{cases}$$

converges strongly as $t \to \infty$ *to a solution of the equation* $A_0 v = 0$.

Remarks: This result can also be derived from a theorem of R. Bruck [6] stating that if $A = \partial\varphi$ for some l.s.c. proper convex function φ and $F \neq \phi$ then for every $u_0 \in \overline{D(A)}$ the solution of the initial value problem

$$(3.9)\qquad \begin{cases} u_t + Au \ni 0 \\ u(0) = u_0 \end{cases}$$

converges weakly as $t \to \infty$ to some solution of $A_0 v = 0$. Using the compactness of the sets K_C defined by (3.7) one sees easily that the convergence of the solutions of the initial value problem (3.8) is actually strong. The strong convergence of the solutions of the initial value problem (3.8) was first proved by H. Brezis in [3] using estimates on the decay of the derivative u_t of the problem (3.9) in conjunction with the compactness of the level sets K_C.

For the problem (3.8) it is not difficult to characterize the set of possible limits of solutions $F_{A_0} = A_0^{-1}0$. Indeed, F_{A_0} is the set of all constant functions $u(x) = \mu$ where $\mu \in \beta^{-1}(0)$.

Next we perturb the problem (3.8) as follows. Let B be a bounded linear operator on $L^2(\Omega)$ satisfying

$$\|B\| = 1 \tag{3.10}$$

and

$$B\cdot 1 = -1 \ . \tag{3.11}$$

Clearly the operator $I + B$ is monotone and since it is everywhere defined and continuous it is maximal monotone. Moreover, by a standard perturbation theorem (see e.g. [2], corollary 2.7) $A_0 + I + B$ is maximal monotone. From (3.11) it follows that F_{I+B} contains all constant functions $u(x) = \alpha$ and therefore

$$F_{I+B} \supset F_{A_0} \ . \tag{3.12}$$

Invoking Theorem 2.6 we obtain

<u>Proposition 3.4</u>:

For every $u_0 \in D(A_0)$ the solution of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u + Bu = 0 & in \ \ \Omega \times (0,\infty) \\ -\dfrac{\partial u}{\partial n} \in \beta(u) & \text{on} \ \ \partial\Omega \times (0,\infty) \\ u(0) = u_0 & \end{cases} \tag{3.13}$$

converges in $L^2(\Omega)$ as $t \to \infty$ to a constant $\alpha(u_0)$ satisfying $\alpha(u_0) \in \beta^{-1}(0)$.

<u>Example 3.5</u>:

Let Ω, H and β be as above and assume again $0 \in \beta(0)$. Let A_1 be defined as follows:

$$D(A_1) = \{u : u \in H^2(\Omega) \cap H_0^1(\Omega),\ \beta(u) \in L^2(\Omega)\} \tag{3.14}$$

and

$$A_1 u = -\Delta u + \beta(u) \quad \text{for} \quad u \in D(A_1) \ . \tag{3.15}$$

It is well known, see e.g. [3] that $A_1 = \partial\varphi_1$ where φ_1 is a proper convex l.s.c. function given by

$$
(3.16) \qquad \varphi_1(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \int_\Omega j(u)dx & \text{for } u \in H_0^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise .} \end{cases}
$$

Again, it follows easily from Rellich's compactness theorem that the sets

$$
K_C = \{u : u \in L^2(\Omega),\ \|u\| \le C,\ \varphi_1(u) \le C\}
$$

are precompact in $L^2(\Omega)$ for every real C. The set $A_1^{-1}0$ is in this case the singleton $\{0\}$ and from Proposition 3.1 we deduce that all the solutions of the initial value problem

$$
(3.17) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + \beta(u) \ni 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(0) = u_0 \end{cases}
$$

converge strongly to zero as $t \to \infty$.

Consider now the first order operator

$$
(3.18) \qquad Lu = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}
$$

defined say on $H_0^1(\Omega)$. L is clearly monotone and $A_1 + L$ is maximal monotone. This follows from a perturbation theorem for maximal monotone operators ([7], theorem 4.4) and the estimate

$$
(3.19) \qquad \|Lu\| \le \varepsilon\|A_1 u\| + C(\varepsilon)\|u\|
$$

that holds for every $\varepsilon > 0$ and $u \in D(A_1)$ because of the compactness of the embedding of $H^2(\Omega) \cap H_0^1(\Omega)$ in $H_0^1(\Omega)$. The estimate (3.19) also shows that A_1 and L are relatively locally bounded since $u \in D(A_1)$, $\|u\| \le R$, $\|(A_1 + L)u\| \le R$ imply

$$
(3.20) \qquad \|A_1 u\| \le \|(A_1 + L)u\| + \|Lu\| \le R + \varepsilon\|A_1 u\| + C(\varepsilon) \cdot R
$$

which implies that $\|A_1 u\|$ is bounded.

Since clearly $0 \in F_L = L^{-1}0$, $F_L \supset F_{A_1}$ and we can apply Theorem 2.6 to obtain;

Proposition 3.6:

For every $u_0 \in D(A_1)$ the solution of the initial value problem

$$
(3.21) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + \beta(u) + Lu \ni 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(0) = u_0 \end{cases}
$$

converges in $L^2(\Omega)$ to zero as $t \to \infty$.

§4. Some remarks concerning the rate of convergence

In Section 2 we saw that if a maximal monotone operator A satisfies the convergence condition (see Definition 2.1) then all the trajectories of the semigroup $S(t)$ generated by A converge strongly to fixed points of $S(t)$. Nothing however, can be said, in general, on the rate of this convergence or on the identification of the limit point in terms of the initial data.

In this section we shall study some special cases of semigroups for which the rate of convergence of the trajectories to fixed points of $S(t)$ can be determined.

In order to study the rate of convergence we introduce gauge functions as follows:

Definition 4.1:

A function $\rho : [0,\infty) \to [0,\infty)$ is called a gauge function if $\rho(s) > 0$ for $s > 0$ and $\lim\limits_{n\to\infty} \rho(s_n) = 0$ implies $\liminf\limits_{n\to\infty} s_n = 0$.

In terms of gauge functions the convergence condition (Definition 2.1) can be restated as follows. *A maximal monotone operator* A satisfies the convergence condition if $F = A^{-1}0 \neq \phi$ and for every constant $C \geq 0$ there is a gauge function $\rho_C(s)$ satisfying

$$(y,x - Px) \geq \rho_C(|x - Px|) \tag{4.1}$$

for all $[x,y] \in A$ such that $|x| \leq C$ and $|y| \leq C$. The gauge functions $\rho_C(s)$, for an operator satisfying the convergence condition are given by

$$\rho_C(s) = \mathrm{Inf}\{(y,x - Px) : [x,y] \in A,\ |x| \leq C,\ |y| \leq C,\ |x - Px| = s\} . \tag{4.2}$$

In some applications the gauge functions $\rho_C(s)$ given in (4.2) are independent of C. In this case we have,

Definition 4.2:

A maximal monotone operator A satisfies the uniform convergence condition if $F = A^{-1}0 \neq \phi$ and there exists a gauge function $\zeta(s)$ such that

$$(y,x - Px) \geq \rho(|x - Px|) \quad \forall [x,y] \in A . \tag{4.3}$$

Example 4.3:

Let A be a strongly monotone operator i.e. there is a positive constant α such that

$$(y_1 - y_2, x_1 - x_2) \geq \alpha|x_1 - x_2|^2 \quad \forall [x_i,y_i] \in A . \tag{4.4}$$

If $F \neq \phi$ we can replace x_2 by Px_1 to obtain (4.3) with the gauge function $\rho(s) = \alpha s^2$. Note that it follows from (4.4) that $A^{-1}0$ contains at most one point. A concrete example which is strongly monotone is given by $A = -\Delta$ with Dirichlet boundary conditions on a bounded domain Ω. This follows from:

$$- \int_\Omega \Delta u \cdot u \, dx = \int_\Omega |\nabla u|^2 dx \geq \gamma_1 \int_\Omega |u|^2 dx$$

where the last inequality is Poincaré's inequality.

Example 4.4:

Let A be a maximal monotone set such that $F = A^{-1}0 \neq \phi$. Assume further that there is a $p \in F$ such that $0 \in \text{int } Ap$. Let $B_\rho(0) = \{x : |x| \leq \rho\} \subset Ap$ then

$$(y - \rho u, x - p) \geq 0 \quad \forall [x,y] \in A \text{ and all } u \text{ satisfying } |u| \leq \rho .$$

Therefore,

$$(y, x - p) \geq \rho |x - p| . \tag{4.5}$$

It follows from (4.5) that $F = \{p\}$ and therefore A satisfies the uniform convergence condition with $\rho(s) = \rho s$.

Example 4.5:

Let A be a linear maximal monotone operator which is σ-angle bounded i.e. there is a constant $\sigma > 0$ such that

$$(Ax,y) \leq \sigma(Ax,x) + (Ay,y) \quad \text{for all } x,y \in D(A) .$$

If the range of A, $R(A)$, is closed then A satisfies the uniform convergence condition. Indeed, from the closed graph theorem it follows that there is a positive constant $\alpha > 0$ such that

$$|Ax| \geq \alpha |x - Px| \quad \text{for all } x \in D(A) , \tag{4.6}$$

where P is the orthogonal projection on $N(A) = F = A^{-1}0$. From the angle boundedness of A it follows that

$$(Ay,x) \leq 2\sqrt{\sigma}\, (Ax,x)^{1/2}(Ay,y)^{1/2} \quad \text{for all } x,y \in D(A) .$$

Now, since $R(A)$ is closed $H = R(A) \oplus N(A)$ and A restricted to $R(A) \cap D(A)$ is one to one and maps $R(A) \cap D(A)$ onto $R(A)$. Let $x \in D(A)$ then $x - Px \in R(A)$ and there is a $y \in R(A) \cap D(A)$ such that $x - Px = Ay$. Since $y \in R(A)$, $Py = 0$ and we have

$$|x - Px|^2 = (Ay, x - Px) \le 2\sqrt{\sigma}\,(Ax, x - Px)^{1/2}(y, Ay)^{1/2} \le 2\sqrt{\sigma}\,|x - Px|^{1/2}|y|^{1/2}(Ax, x - Px)^{1/2}.$$

But from (4.6) it follows that $|Ay| \ge \alpha|y|$ and therefore

$$|x - Px|^2 \le 2\sqrt{\frac{\sigma}{\alpha}}\,|x - Px|\,(Ax, x - Px)^{1/2}$$

which implies

$$(4.7) \qquad (Ax, x - Px) \ge \frac{\alpha}{4\sigma}\,|x - Px|^2 .$$

Note that if A is self adjoint then it is angle bounded and $\sigma = 1/4$. So if A is self adjoint and $R(A)$ is closed A satisfies the uniform convergence condition.

On the other hand it is well known that if A is a positive self adjoint operator then $S(t)x \to p$ as $t \to \infty$ for every $x \in H$. Since it is rather easy to exhibit a positive self adjoint operator that does not satisfy the convergence condition we see, as we have already mentioned in the introduction, that the convergence condition is not necessary for the strong convergence of $S(t)x$ as $t \to \infty$.

A simple example of a self adjoint positive bounded operator that does not satisfy the convergence condition is given in the real ℓ^2 space by $Ax = \{n^{-1}\xi_n\}$ where $x = \{\xi_n\}_{n=1}^{\infty}$. It is easy to check that $N(A) = \{0\}$ and that if we denote by $e_n = \{\delta_{nk}\}_{k=1}^{\infty}$ we have $(Ae_n, e_n) \to 0$, $|e_n| = 1$, $|Ae_n| \le 1$ but $\{e_n\}_{n=1}^{\infty}$ has no subsequence that converges strongly to zero.

When A satisfies the uniform convergence condition we can often estimate the rate of convergence of $S(t)x$ to its limit as $t \to \infty$. Two typical examples will be given below. We start with the following lemma.

Lemma 4.6:

If A satisfies the convergence condition and $p = \lim_{t\to\infty} PS(t)x$ then

$$(4.8) \qquad |S(t)x - p| \le 2|S(t)x - PS(t)x| \qquad \forall x \in \overline{D(A)} .$$

Proof: We have already proved in Theorem 2.2 that if A satisfies the convergence condition then

$$\lim_{t\to\infty} S(t)x = \lim_{t\to\infty} PS(t)x = p .$$

From the proof of Lemma 2.3 it follows that if $s \geq t$ then

$$|PS(s)x - PS(t)x|^2 \leq |S(t)x - PS(t)x|^2 - |S(s)x - PS(s)x|^2 .$$

Letting $s \to \infty$ we obtain

$$|p - PS(t)x|^2 \leq |S(t)x - PS(t)x|^2 \tag{4.9}$$

and therefore,

$$|S(t)x - p| \leq |S(t)x - PS(t)x| + |PS(t)x - p| \leq 2|S(t)x - PS(t)x| . \quad \blacksquare$$

Remark: Letting $t \to 0$ in (4.9) yields

$$|p - Px| \leq |x - Px| \tag{4.10}$$

which gives us some information on the location of the limit p. Simple examples in $\mathbb{R}^2$ show that we may have equality in (4.10).

Proposition 4.7:

Let A be maximal monotone. If A satisfies the uniform convergence condition with a gauge function $\rho(s)$ such that $\rho(s) \geq cs^2$ then the convergence of $S(t)x$ to a fixed point has exponential rate, i.e. for every $x \in \overline{D(A)}$ there is a $p \in F$ such that

$$|S(t)x - p| \leq 2e^{-ct}|x - Px| . \tag{4.11}$$

Proof: We assume first that $x \in D(A)$. Multiplying the equation

$$u_t + Au \ni 0$$

by $u - Pu$ we obtain

$$\frac{1}{2}\frac{d}{dt}|u - Pu|^2 + c|u - Pu|^2 \leq 0$$

which implies

$$|u - Pu| \leq e^{-ct}|x - Px| \tag{4.12}$$

and the result now follows from Lemma 4.6. The result for $x \in \overline{D(A)}$ is obtained by continuity. $\blacksquare$

Assuming some stronger conditions on the gauge function $\rho(s)$ we obtain convergence to the limit in "finite time".

Proposition 4.8:

Let A be maximal monotone satisfying the uniform convergence condition with a gauge function $\rho(s)$. If $\rho(s)$ is measurable and for every $C > 0$

$$\int_0^c \frac{s}{\rho(s)}\,ds < \infty \tag{4.13}$$

then for every $x \in \overline{D(A)}$ there exists a T_x, $0 \le T_x < \infty$ and a $p \in F$ such that $S(t)x = p$ for all $t \ge T_x$. Moreover

$$T_x \le \int_0^{|x-Px|} \frac{s}{\rho(s)}\,ds \,. \tag{4.14}$$

Proof: We start again with $x \in D(A)$ and obtain the result for every $x \in \overline{D(A)}$ by continuity. From the uniqueness of the solution of the initial value problem

$$\begin{cases} u_t + Au \ni 0 \\ u(0) = x \end{cases} \tag{4.15}$$

it follows that if for some $0 \le t_0 < \infty$, $u(t_0) = p \in F$ then $u(t) = p$ for all $t \ge t_0$. We define:

$$G(r) = \int_0^r \frac{s}{\rho(s)}\,ds \,. \tag{4.16}$$

It follows from (4.13) that $G(r)$ is well defined and satisfies $G(r) > 0$ for $r > 0$. Let $s(t) = |u(t) - Pu(t)|$. Multiplying (4.15) by $u - Pu$ we obtain

$$\frac{1}{2}\frac{d}{dt}\, s(t)^2 + \rho(s(t)) \le 0 \,. \tag{4.17}$$

From (4.17) it follows that $s(t)$ is nonincreasing. We assume that $s(0) = |x - Px| > 0$. Suppose $s(t) > 0$ for all t, $0 \le t \le T$. Dividing (4.17) by $\rho(s(t))$ and integrating from 0 to T yields

$$G(s(T)) - G(s(0)) + T \le 0$$

and hence

$$T \le G(s(0)) \,. \tag{4.18}$$

Therefore, $s(t)$ must vanish for a finite T_x satisfying $T_x \le G(|x - Px|)$. ∎

Corollary 4.9:

If A satisfies the uniform convergence condition with a gauge function $\rho(s)$ satisfying $\rho(s) \ge cs^{\alpha}$ with $0 < \alpha < 2$ then for each $x \in \overline{D(A)}$ there is a $p \in F$ such that $S(t)x = p$ for all $t \ge c^{-1}(2 - \alpha)^{-1}|x - Px|^{2-\alpha}$.

If A satisfies the uniform convergence condition with a gauge function $\rho_A(s)$ then the perturbation theorem (Theorem 2.6) reduces to the following simpler statement.

<u>Proposition 4.10</u>:

Let A be maximal monotone and let B be monotone such that $A + B$ is maximal monotone. If A satisifes the uniform convergence condition with a gauge function $\rho_A(s)$ and $F_B \supset F_A$ then $A + B$ satisfies the uniform convergence condition with the same gauge function.

<u>Proof</u>: As in the proof of Theorem 2.6 we show that $F_{A+B} = F_A$ and therefore the projection P_{A+B} on F_{A+B} is the same as P_A, the projection on F_A. If $[x,y] \in A + B$ then $y = \eta_1 + \eta_2$ with $\eta_1 \in Ax$, $\eta_2 \in Bx$ and

$$(y, x - P_{A+B}x) = (\eta_1, x - P_Ax) + (\eta_2, x - P_Ax) \geq (\eta_1, x - P_Ax) \geq \rho_A(|x - P_{A+B}x|)$$

and the proof is complete. ∎

<u>Example 4.11</u>:

Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $H = L^2(\Omega)$ and let A be any maximal monotone operator on H satisfying $0 \in A0$. Let $\alpha > -2$ and let B be the everywhere defined operator

$$Bu = u \cdot (\int_\Omega |u|^2 dx)^{\alpha/2} = u \cdot \|u\|^\alpha . \tag{4.19}$$

It is not difficult to see that $B \subset \partial\varphi$ where

$$\varphi(u) = (\alpha + 2)^{-1}\|u\|^{\alpha+2} \tag{4.20}$$

and therefore B is monotone. Since B is continuous and everywhere defined it is maximal monotone and $B = \partial\varphi$. Moreover, $A + B$ is maximal monotone (see e.g. [2], corollary 2.7) and for every $u \in D(A)$ the initial value problem

$$\begin{cases} \dfrac{du}{dt} - Au + u\|u\|^\alpha \ni 0 \\ u(0) = u_0 \end{cases} \tag{4.21}$$

has a strong solution. Clearly $0 \in F_A$ and $F_B = \{0\}$ so $F_A \supset F_B$. Also

(4.22) $$(Bu,u - P_B u) = (Bu,u) = \left(\int_\Omega |u|^2 dx\right)^{\frac{\alpha}{2}+1} = ||u||^{\alpha+2}$$

and therefore B satisfies the *uniform convergence condition with gauge function* $\rho(s) = s^{\alpha+2}$. Thus by Theorem 4.10, $A + B$ satisfies the uniform convergence condition with the same gauge function and we have:

Proposition 4.12:

For every $\alpha > -2$ the solution of the initial value problem (4.21) converges strongly to zero as $t \to \infty$. For $-2 < \alpha < 0$ the solution reaches its limit in finite time and for $\alpha = 0$ the convergence is at least exponential.

§5. The inhomogeneous equation

In this section we study the asymptotic behavior as $t \to \infty$ of solutions of the initial value problem

$$(5.1) \qquad \begin{cases} \frac{du}{dt} + Au \ni f(t) \\ u(0) = x \end{cases} .$$

By a solution of (5.1) we mean a weak solution in the sense of [2], (Chapter 3, definition 3.1). It was proved in [2], [4] that if $f \in L^1(0,T;H)$ then for every $x \in H$, (5.1) has a unique solution $u(t) \in C([0,T];H)$. Moreover, if $v(t)$ is a solution of the initial value problem (5.1) with $f(t)$ replaced by $g(t)$ then

$$(5.2) \qquad |u(t) - v(t)| \le |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)|d\sigma, \qquad 0 \le s \le t \le T .$$

In Section 2 we saw that if A satisfies the convergence condition and $f(t) \equiv 0$ then the solution of (5.1) converges strongly as $t \to \infty$ to a solution p of $0 \in Ap$. Our next result shows that this stays true if instead of $f(t) = 0$ we have $f(t) \in L^1(0,\infty;H)$ and the solution of (5.1) is interpreted in the weak sense.

Theorem 5.1:

Let A be maximal monotone satisfying the convergence condition. If $f \in L^1(0,\infty;H)$ then for every $x \in H$ the solution of (5.1) converges strongly as $t \to \infty$ to a point $p \in F = A^{-1}0$.

Proof: Let $u(t)$ be the solution of (5.1). Given any $\varepsilon > 0$ let T be chosen so that

$$(5.3) \qquad \int_T^\infty |f(\sigma)|d\sigma < \varepsilon$$

and let $v(t)$ be the solution of

$$(5.4) \qquad \begin{cases} \frac{dv}{dt} + Av \ni 0 \\ v(0) = u(T) \end{cases} .$$

For $t > T$ we then have by (5.2)

$$(5.5) \qquad |u(t) - v(t - T)| \le \int_T^t |f(\sigma)|d\sigma < \varepsilon .$$

Hence for every $t,s \geq T$

(5.6) $$|u(t) - u(s)| < 2\varepsilon + |v(t-T) - v(s-T)| .$$

But from Theorem 2.2 we know that $v(t) \to p$ as $t \to \infty$ and therefore by choosing s and t large enough the second term on the right of (5.6) can be made as small as we wish. Consequently, $u(t)$ is a Cauchy net and we have $u(t) \to u_\infty$ as $t \to \infty$. Since F is closed, it follows from (5.5) that $u_\infty \in F$. ■

Remark 5.2:

The previous theorem shows that the solutions of the initial value problem (5.1) and the initial value problem

(5.7) $$\begin{cases} \dfrac{du}{dt} + Au \ni 0 \\ u(0) = x \end{cases}$$

both converge as $t \to \infty$ to elements of F. Clearly, in general, the limits are not the same and moreover the rate of convergence to the limit may be drastically changed when we pass from (5.7) to (5.1). For example, if A satisfies the uniform convergence condition with gauge function $\rho(s)$ and $\rho(s)$ satisfies (4.13) then the solution of (5.7) reaches its limit in finite time, whereas the solution of (5.1) cannot converge to its limit in finite time unless $f(t)$ has compact support.

Given any $g \in H$ we define an operator A_g by, $D(A_g) = D(A)$ and

(5.8) $$A_g x = Ax - g \quad \forall x \in D(A) .$$

Clearly A_g is maximal monotone if and only if A is maximal monotone and the initial value problem (5.1) is equivalent to the problem

(5.9) $$\begin{cases} \dfrac{du}{dt} + A_g u \ni f(t) - g \\ u(0) = x \end{cases} .$$

From this observation and Theorem 5.1 we obtain

Theorem 5.3:

Consider the initial value problem (5.1). If there exists an element $f_\infty \in H$ such that $f(t) - f_\infty \in L^1(0,\infty;H)$ and the operator A_{f_∞} satisfies the convergence condition then for every $x \in H$ the solution $u(t)$ of (5.1) converges strongly to an element $p \in A^{-1} f_\infty$.

Usually, one cannot obtain information about whether or not A_f satisfies the convergence condition from the fact that A satisfies this condition. In some cases however this is possible. We start with the following proposition.

Proposition 5.4:

Let A be maximal monotone satisfying

$$(y_1 - y_2, x_1 - x_2) > 0 \quad \forall [x_i, y_i] \in A \quad i = 1,2, \text{ such that } x_1 \neq x_2 . \tag{5.10}$$

If $(I + A)^{-1}$ is compact then for every $f \in R(A)$, A_f satisfies the convergence condition.

Proof: Clearly $f \in R(A)$ implies $A^{-1}f = A_f^{-1}0 \neq \phi$. Also $(I + A)^{-1}$ compact implies that $(I + A_f)^{-1}$ is compact since

$$(I + A)^{-1}x = (I + A_f)^{-1}(x - f) . \tag{5.11}$$

From the assumption (5.10) we have

$$(y - f, x - \tilde{P}x) > 0 \quad \forall [x,y] \in A \text{ s.t. } x \neq \tilde{P}x$$

where $\tilde{P}$ is the projection on $A^{-1}f$. Therefore

$$(z, x - \tilde{P}x) > 0 \quad \forall [x,z] \in A_f$$

and the result follows from Proposition 3.2. ∎

A somewhat similar situation holds in the case where $A = \partial\varphi$ and φ has compact level sets. In this case we have,

Proposition 5.5:

Let φ be a proper convex l.s.c. function for which the level sets

$$K(R_1, R_2) = \{x : |x| \leq R_1 \quad \varphi(x) \leq R_2\} \tag{5.12}$$

are precompact for all $R_1 \geq 0$ and R_2 real. If $A = \partial\varphi$ then for every $f \in R(A)$, A_f satisfies the convergence condition.

Proof: Since $f \in R(A)$, $A_f^{-1}0 \neq \phi$. Let $f \in A\xi$ then

$$\varphi(x) - (f,x) \geq \varphi(\xi) - (f,\xi) \quad \forall x \in H .$$

Therefore

$$\gamma = \operatorname*{Min}_{x \in H}\{\varphi(x) - (f,x)\} = \varphi(\xi) - (f,\xi) > -\infty .$$

Let

$$\psi(x) = \varphi(x) - (f,x) - \gamma . \tag{5.13}$$

Clearly $\psi(x)$ is proper convex, lower semicontinuous, $\psi(x) \geq 0$ and $\min_{x \in H} \psi(x) = 0$. Moreover $\partial\psi = \partial\varphi - f = A_f$. In order to show that A_f satisfies the convergence condition it suffices (by Proposition 3.1) to show that the level sets of $\psi(x)$ are precompact but this is obvious in view of the definition of $\psi(x)$ and our hypothesis (5.12). ∎

From Proposition 5.5 we immediately obtain the following result of H. Brezis ([2], Chapter 3, theorem 3.11).

Corollary 5.6:

Let φ be a proper convex l.s.c. function such that the level sets of φ

$$K(R_1,R_2) = \{x : |x| \leq R_1, \varphi(x) \leq R_2\}$$

are precompact for every $R_1 \geq 0$ and R_2 real. Let $A = \partial\varphi$, $f_\infty \in R(A)$ and let $f(t)$ be a function satisfying $f(t) - f_\infty \in L^1(0,\infty;H)$. If $u(t)$ is the solution of (5.1) then $u(t)$ converges strongly as $t \to \infty$ to a limit $u_\infty \in A^{-1}f_\infty$.

§6. A discrete version of the convergence theorem

In this section we discuss briefly a discrete version of Theorems 2.2 and 5.1. This version consists of replacing the differential equation defining $S(t)x$ by the finite difference scheme

$$(6.1) \qquad \begin{cases} \dfrac{x_n - x_{n-1}}{\lambda_n} + Ax_n \ni 0 \\ x_0 = x \end{cases}$$

or equivalently

$$(6.2) \qquad x_n = J_{\lambda_n} x_{n-1}, \; x_0 = x$$

and studying the convergence of x_n as $n \to \infty$. We shall see that if A satisfies the convergence condition and $\sum_{n=1}^{\infty} \lambda_n = \infty$ then $x_n \to p \in F$ as $n \to \infty$. Actually, we shall allow for some errors in the difference scheme (6.1) and rather than defining the sequence $\{x_n\}$ by (6.2) we shall define it by

$$(6.3) \qquad x_n = J_{\lambda_n}(x_{n-1} + e_n), \quad x_0 = x$$

where e_n is an error term. From (6.3) we have

$$(6.4) \qquad x_n + \lambda_n y_n = x_{n-1} + e_n, \quad [x_n, y_n] \in A$$

and we shall henceforth, in this section, denote the element $y_n \in Ax_n$ defined by (6.4) by Ax_n.

We start with some general properties of the sequence $\{x_n\}$ *defined by* (6.3).

Proposition 6.1:

Let A *be maximal monotone with* $A^{-1}0 \neq \phi$ *and let* P *be the projection on* $F = A^{-1}0$. *Let* $\{x_n\}$ *be the sequence defined by* (6.3). *If* $\sum_{n=1}^{\infty} |e_n| < \infty$ *then*

i) $|x_n - Px_n| \leq M$ for all n.

ii) $\lim_{n\to\infty} |x_n - Px_n|$ exists.

iii) $\lim_{n\to\infty} Px_n = p \in F$ exists.

iv) *If furthermore* $\omega_w(x) \subseteq F$ *then* $x_n \rightharpoonup p \in F$ *as* $n \to \infty$. *Here* $\rightharpoonup$ *means weak convergence in* H *and* $\omega_w(x)$ *is the weak* ω*-limit set of* x *i.e.*

$$\omega_w(x) = \{y : \exists n_k \to \infty \text{ such that } x_{n_k} \rightharpoonup y\} .$$

Proof: We have

$$(6.5)\qquad |x_n - Px_n| \le |x_n - Px_{n-1}| = |J_{\lambda_n}(x_{n-1} + e_n) - Px_{n-1}| \le |x_{n-1} - Px_{n-1}| + |e_n| \,.$$

Iterating (6.5) between k and $n \ge k$ we obtain

$$(6.6)\qquad |x_n - Px_n| \le |x_k - Px_k| + \sum_{j=k+1}^{n} |e_j|$$

which implies (i). Taking the upper limit on n in (6.6) yields

$$(6.7)\qquad \limsup_{n\to\infty} |x_n - Px_n| \le |x_k - Px_k| + \sum_{j=k+1}^{\infty} |e_j| \quad \forall k \,.$$

Taking now the lower limit on the right hand side of (6.7) proves (ii). To prove (iii) note that from the definition of P it follows that

$$(6.8)\qquad |Px_n - Px_k|^2 \le |x_n - Px_k|^2 - |x_n - Px_n|^2 \,.$$

Also,

$$(6.9)\qquad |x_n - Px_k| = |J_{\lambda_n}(x_{n-1} + e_n) - Px_k| \le |x_{n-1} - Px_k| + |e_n| \,.$$

Iterating (6.9) down to $n = k$ we obtain

$$|x_n - Px_k| \le |x_k - Px_k| + \sum_{j=k+1}^{n} |e_j|$$

and therefore

$$(6.10)\qquad |x_n - Px_k|^2 \le |x_k - Px_k|^2 + 2M \sum_{j=k+1}^{n} |e_j| + \Big(\sum_{j=k+1}^{n} |e_j|\Big)^2 \,.$$

Substituting (6.10) into (6.8) and using (ii) it follows that Px_n is a Cauchy sequence and therefore converges to some $p \in F$.

Finally, to prove (iv), note that from (i) and (iii) it follows that $\{x_n\}$ is bounded. Let $x_{n_k} \rightharpoonup \ell$. From our condition it follows that $\ell \in F$. By the definition of P we have

$$(6.11)\qquad (x_{n_k} - Px_{n_k}, \ell - Px_{n_k}) \le 0 \,.$$

Letting $n_k \to \infty$ we conclude that $\ell = p = \lim_{n\to\infty} Px_n$. Since the limit is independent of the sequence $\{n_k\}$ it follows that $x_n \rightharpoonup p$ as $n \to \infty$. ∎

Remark: If we define $\{x_n\}$ by (6.2) i.e. with $e_n = 0$ for all n, it was shown by H. Brezis and P. L. Lions that if $\Sigma \lambda_n^2 = \infty$, $Ax_n \to 0$ for any initial $x \in H$ and therefore $x_n \rightharpoonup p \in F$. Since clearly $Ax_n \to 0$ implies $\omega_w(x) \subseteq F$ this result also follows from Proposition 6.1 once we know that $Ax_n \to 0$. H. Brezis and P. L. Lions [5] also showed that if $A = \partial\varphi$ then $\sum_{n=1}^{\infty} \lambda_n = \infty$ implies $Ax_n \to 0$ and therefore $x_n \rightharpoonup p$.

In the linear case part (iv) of Proposition 6.1 can be improved as follows.

Proposition 6.2:

Let A be linear maximal monotone. If $\forall x \in H$ we have for the sequence $\{x_n\}$ defined by (6.2) $Ax_n \to 0$ then $\forall x \in H$, $x_n \to Px$ where P is the orthogonal projection on $F = N(A)$.

Proof: If A is linear $A0 = 0$ and therefore $F \neq \phi$. Also for maximal monotone A we have $H = N(A) \oplus \overline{R(A)}$ where $\overline{R(A)}$ is the closure of the range of A. This implies that for every $\varepsilon > 0$ there are $y_0 \in D(A)$ and r_0 such that

$$x - Px = Ay_0 + r_0 \qquad |r_0| < \varepsilon . \tag{6.12}$$

Starting the iteration with (6.12) and noting that $J_{\lambda_n} Px = Px$ we obtain

$$x_n - Px = Ay_n + r_n \tag{6.13}$$

where $y_n = J_{\lambda_n} y_{n-1}$ and $r_n = J_{\lambda_n} r_{n-1}$. Since by our assumption $Ay_n \to 0$ (6.13) implies

$$\limsup_{n\to\infty} |x_n - Px| \leq \varepsilon$$

and since $\varepsilon > 0$ is arbitrary $x_n \to Px$ as $n \to \infty$. ∎

We now turn to the discrete analogue of Theorem 5.1.

Theorem 6.3:

Let A be maximal monotone satisfying the convergence condition. Let $x_0 = x \in H$ and

$$x_n = J_{\lambda_n}(x_{n-1} + e_n), \qquad n = 1,2,\ldots .$$

If $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |e_n| < \infty$ then x_n converges strongly as $n \to \infty$ to some $p \in F$.

Proof: From the definition of x_n we have

(6.14) $$x_n - Px_n + \lambda_n Ax_n = x_{n-1} - Px_n + e_n .$$

Multiplying (6.14) by $x_n - Px_n$ and rearranging we obtain

(6.15) $$\lambda_n(Ax_n, x_n - Px_n) \leq |x_{n-1} - Px_{n-1}|^2 - |x_n - Px_n|^2 + (e_n, x_n - Px_n) + (Px_{n-1} - Px_n, x_n - Px_n) .$$

By the definition of P the last term on the right of (6.15) is nonpositive and therefore,

$$\lambda_n(Ax_n, x_n - Px_n) \leq |x_{n-1} - Px_{n-1}|^2 - |x_n - Px_n|^2 + M|e_n|$$

which implies

(6.16) $$\sum_{n=1}^{\infty} \lambda_n(Ax_n, x_n - Px_n) < \infty .$$

Since by assumption $\sum_{n=1}^{\infty} \lambda_n = \infty$ it follows from (6.16) that

$$\liminf_{n\to\infty} (Ax_n, x_n - Px_n) = 0$$

and therefore from the convergence condition

$$\liminf_{n\to\infty} |x_n - Px_n| = 0 .$$

From Proposition 6.1 (ii) we deduce that $|x_n - Px_n| \to 0$ and since $Px_n \to p \in F$ also $x_n \to p \in F$ and the proof is complete. ∎

Remark: In the continuous case we saw that if A satisfies the convergence condition with a gauge function $\rho(s)$ that satisfies (4.13) then $S(t)x$ converges to its limit in finite time. This "finite time" convergence is very delicate and as we saw in Section 5 once the equation

(6.17) $$\begin{cases} \dfrac{du}{dt} + A^0 u = 0 \\ u(0) = x \end{cases}$$

is satisfied only approximately i.e. there is an error term which is $L^1(0,\infty;H)$ one can no more have convergence in finite time. The discrete sequence

(6.18) $$x_n = J_{\lambda_n} x_{n-1}, \quad x_0 = x$$

is just an approximation to (6.17) and therefore one cannot expect convergence in finite time for x_n, and indeed this is impossible in the discrete case since $x_n = p \in F$ implies $x_{n-1} = p \in F$ and therefore $x = p \in F$.

On the other hand if we consider the sequence

$$x_n = J_{\lambda_n}(x_{n-1} + e_n), \quad x_0 = x \tag{6.19}$$

then one can hit a fixed point p in finite time, but unless all $e_n = 0$ after this n, the fixed point p will not necessarily be the limit of the sequence $\{x_n\}$ and clearly we will not have $x_n \equiv p$ for $n \geq N$.

In the case A satisfies the uniform convergence condition one can often deduce also in the discrete case a rate of convergence of x_n to p. The case where all $e_n = 0$ is the simplest. In this case $|x_n - Px_n|$ is monotonically nonincreasing and from (6.10) passing to the limit as $n \to \infty$ we obtain $|p - Px_k| \leq |x_k - Px_k|$ and therefore

$$|p - x_n| \leq 2|x_n - Px_n| \ . \tag{6.20}$$

Assuming now that A satisfies the uniform convergence condition with a strictly increasing gauge function $\rho(s)$ one deduces from (6.16) that

$$\sum_{n=1}^{\infty} \lambda_n \rho(|x_n - Px_n|) \leq |x - Px|^2 \tag{6.21}$$

which implies

$$|x_n - p| \leq 2\rho^{-1}\left(\frac{|x - Px|^2}{\sum_{k=1}^{n} \lambda_k}\right) . \tag{6.22}$$

This is a much slower rate of convergence than the rate that was obtained in the corresponding continuous case.

§7. A nonlinear Neumann problem

In this section we consider a concrete example of a nonlinear Neumann problem and study its asymptotic behavior. In the study of this example we use some of the results that were developed in the previous sections as well as some ideas motivated by these results.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The measure of Ω will be denoted by $|\Omega|$. Let H be the Hilbert space $L^2(\Omega)$ with the norm denoted by $\|\cdot\|$ and let β be a maximal monotone graph with primitive j, satisfying $0 \in \beta(0)$. Consider the initial value problem

$$(7.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \beta(u) \ni 0 & \text{in} \quad \Omega \times [0,\infty) \\[2ex] \dfrac{\partial u}{\partial n} = 0 & \text{in} \quad \partial\Omega \times [0,\infty) \\[2ex] u(0,x) = u_0(x) & \text{in} \quad \Omega \ . \end{cases}$$

It is well known that for every $u_0 \in L^2(\Omega)$ the initial value problem (7.1) has a unique solution u satisfying $u \in C([0,\infty); L^2(\Omega))$ and $u(t) \in H^2(\Omega)$ for every $t > 0$ (see e.g. [3]).

Moreover, the operator $A = -\Delta + \beta$ with the Neumann boundary conditions appearing in the problem is the subdifferential of the lower semicontinuous convex function φ given by

$$(7.2)\qquad \varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega j(u)dx & \text{for } u \in H^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\[2ex] +\infty & \text{otherwise} \ . \end{cases}$$

It follows immediately from Rellich's compactness theorem that the sets

$$K_C = \{u : u \in L^2(\Omega), \|u\| \le C, \varphi(u) \le C\}$$

are precompact in $L^2(\Omega)$ for every real C. In order to use the results of the previous sections we have to identify the set $A^{-1}0$. This we do in the next lemma.

Lemma 7.1:

Let $\beta_1 < 0 < \beta_2$ and assume that $[\beta_1,\beta_2] = \beta^{-1}(0)$. For the operator $A = -\Delta + \beta$ with Neumann boundary conditions, $A^{-1}0$ is the set of all constant functions $u = \gamma$

with $\beta_1 \le \gamma \le \beta_2$. The projection P on this set is given by:

$$Pu = \begin{cases} \beta_2 & \text{if } \bar{u} \ge \beta_2 \\ \bar{u} & \text{if } \beta_1 \le \bar{u} \le \beta_2 \\ \beta_1 & \text{if } \bar{u} \le \beta_1 \end{cases} \tag{7.3}$$

where

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx . \tag{7.4}$$

Proof: Let $Au = 0$. Since $0 \in \beta(0)$ we have $u \cdot \beta(u) \ge 0$ and therefore:

$$0 = (Au,u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega \beta(u) \cdot u dx \ge \int_\Omega |\nabla u|^2 dx . \tag{7.5}$$

This implies $\nabla u = 0$ a.e. and thus $u = \text{const.}$ From (7.5) it then follows that $\beta(u) = 0$ and therefore $\beta_1 \le u \le \beta_2$. To compute P we consider

$$Pu = \operatorname*{Inf}_{\beta_1 \le k \le \beta_2} \int_\Omega |u - k|^2 dx$$

and a simple computation shows that Pu is given by (7.3). ■

Combining the previous lemma with Proposition 3.2 we obtain

Proposition 7.2:

If $\beta_1 < 0 < \beta_2$ and $[\beta_1, \beta_2] = \beta^{-1}(0)$ then for every $u_0 \in L^2(\Omega)$ the solution $u(t,x)$ of (7.1) converges (in $L^2(\Omega)$) as $t \to \infty$ to a constant u_∞ satisfying $\beta_1 \le u_\infty \le \beta_2$.

In the rest of this section we shall study the rate of convergence of $u(t,x)$ to its limit and we shall try to estimate the limit u_∞ as a function of the initial data u_0. We start with the following proposition.

Proposition 7.3:

Let β be a maximal monotone graph satisfying $0 \in \beta(0)$. Let $u(t,x)$ be the solution of the initial value problem (7.1). If

$$\bar{u}(t) = \frac{1}{|\Omega|} \int_\Omega u(t,x) dx$$

then

(7.5) $$\int_\Omega |u(t,x) - \bar{u}(t)|^2 dx \le e^{-2\gamma_1 t} \int_\Omega |u_0(x) - \bar{u}_0|^2 dx$$

where γ_1 is the smallest nonzero eigenvalue of $-\Delta$ with Neumann conditions on $\partial\Omega$.

Proof: Multiplying the equation in (7.1) by $u(t,x) - \bar{u}(t)$ and integrating over Ω yields:

(7.6) $$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \bar{u}|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega \beta(u)(u - \bar{u})dx = 0 .$$

Since $\beta(u)(u - \bar{u}) \ge \beta(\bar{u})(u - \bar{u})$ we have

$$\int_\Omega \beta(u)(u - \bar{u})dx \ge \int_\Omega \beta(\bar{u})(u - \bar{u})dx = 0$$

and therefore,

(7.7) $$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \bar{u}|^2 dx + \int_\Omega |\nabla u|^2 dx \le 0 .$$

Using Poincaré's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \bar{u}|^2 dx + \gamma_1 \int_\Omega |u - \bar{u}|^2 dx \le 0$$

which implies (7.5). ■

If $\beta \equiv 0$ one obtains easily by integrating the equation over Ω that $\bar{u}(t) = \bar{u}_0$ for all $t \ge 0$ and therefore we have:

Corollary 7.4:

If $\beta \equiv 0$ then

(7.8) $$\int_\Omega |u(t,x) - \bar{u}_0|^2 dx \le e^{-2\gamma_1 t} \int_\Omega |u_0 - \bar{u}_0|^2 dx .$$

In the linear case we can therefore identify the limit u_∞ of $u(t,x)$. The limit in this case is $u_\infty = \bar{u}_0$ and the convergence of $u(t,x)$ to u_∞ is exponential. We shall henceforth assume that β is a maximal monotone graph satisfying

(7.9) $$\beta^{-1}(0) = [\beta_1,\beta_2] \text{ where } \beta_1 \le 0 \le \beta_2 .$$

Clearly one cannot expect exponential convergence to the limit for arbitrary β satisfying (7.9). Indeed, other behaviors are exhibited in the next example.

Example 7.5:

Let β satisfy (7.9) and let $u(t,x)$ be the solution of (7.1) with $u(0,x) = u_0$ where u_0 is a constant satisfying $u_0 > \beta_2$. In this case $u(t,x)$ is independent of x and satisfies

$$(7.10)\qquad \begin{cases} \dfrac{du}{dt} + \beta(u) \ni 0 \\ u(0) = u_0 \end{cases} .$$

From Proposition 7.2 we know that $u(t) \to u_\infty$ as $t \to \infty$. The rate of convergence however depends strongly on β. Choosing for example

$$(7.11)\qquad \beta(s) = \begin{cases} 0 & s \le \beta_2 \\ \gamma(s - \beta_2)^k & s \ge \beta_2 \end{cases}$$

we obtain an explicit solution of (7.10) given by

$$(7.12)\qquad u(t) = \beta_2 - (u_0 - \beta_2)[1 + \gamma(k-1)(u_0 - \beta_2)^{k-1}t]^{1/1-k} .$$

If $k > 1$ the convergence of $u(t)$ to β_2 is very slow, while if $k < 1$, $u(t)$ reaches its limit β_2 in finite time.

In the rest of this section we shall assume for simplicity that $u_0(x) \ge 0$ and that β satisfies:

$$(7.13)\qquad \beta(s) \equiv 0 \text{ for } s \le 1 \text{ and } \beta(s) > 0 \text{ for } s > 1 .$$

Heuristically it is clear that if $u_\infty < 1$ the rate of convergence of $u(t,x)$ to u_∞ can be at most exponential with the exponent γ_1 being the smallest nonzero eigenvalue of $-\Delta$ with Neumann boundary conditions on $\partial\Omega$. The reason for this is that for large t, u will be near its limit u_∞ and thus $\beta(u)$ will essentially be zero so that the problem will behave for large values of t as the linear problem. In order to make this heuristic argument rigorous one has to prove that $u(t,x)$ converges to u_∞ in $L^\infty(\Omega)$. This can be done but we shall not do it here.

In order to obtain estimates on the rate of convergence to the limit u_∞ we shall need the following lemma.

Lemma 7.6:

Let $u_0 \in L^2(\Omega)$, $u_0 \geq 0$. The solution of (7.1) satisfies

(7.14) $$u(t,x) \in L^\infty(\Omega) \quad \text{for every} \quad t > 0 .$$

Proof: From the maximum principle it readily follows that $u(t,x) \geq 0$ a.e. for all $t \geq 0$. Consider now the comparison function $v(t,x)$ satisfying

(7.15) $$\begin{cases} v_t - \Delta v = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \dfrac{\partial v}{\partial n} = 0 & \text{in} \quad \partial\Omega \times (0,\infty) \\ v(x,0) = u_0(x) & \text{in} \quad \Omega . \end{cases}$$

From the maximum principle we deduce

(7.16) $$0 \leq u(t,x) \leq v(t,x) \quad \text{in} \quad \Omega \times (0,\infty) .$$

For the linear problem (7.15) however, it follows easily from the fact that $v = G * v_0$ and the known estimates on the Green's function G (see [9]) that $v(t,x) \in L^\infty(\Omega)$ for $t > 0$ and therefore the result follows from (7.16). ∎

In our next proposition we shall use the following terminology. A maximal monotone graph β satisfying (7.13) will be called forceful if it has the following property; for every $\zeta > 1$ the solution of the ordinary differential equation

(7.17) $$\begin{cases} \dfrac{dv}{dt} + \beta(v) \ni 0 \\ v(0) = \zeta \end{cases}$$

reaches the value $v = 1$ in finite time. Examples of forceful β are

(7.18) $$\beta(s) = \begin{cases} 0 & s \leq 1 \\ \gamma(s-1)^\alpha & s \geq 1 \end{cases}$$

where $\gamma > 0$ and $0 \leq \alpha < 1$.

Proposition 7.7:

Let β be forceful. For every $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ there is a $T < \infty$ such that the solution $u(t,x)$ of (7.1) satisfies:

(7.19) $$u(t,x) \leq 1 \quad \text{a.e. for} \quad t \geq T .$$

Moreover, if $u_\infty < 1$ the convergence of $u(t,x)$ to u_∞ has exponential rate, whereas if $u_\infty = 1$ the limit is reached in finite time.

Proof: From Lemma 7.6 it follows that without loss of generality we can assume that $u_0 \in L^\infty(\Omega)$. Assuming this, it follows from the maximum principle that $0 \le u(t,x) \le v(t)$ where $v(t)$ is the solution of (7.17) with $\zeta = \|u_0\|_{L^\infty(\Omega)}$. If $\|u_0\|_{L^\infty} < 1$ we are in the linear situation and by Corollary 7.4 we have $u_\infty = \bar{u}_0$ and the rate of convergence is exponential. If $\|u_0\|_{L^\infty} > 1$ it follows from our assumptions on β that $v(t)$ reaches the value 1 in finite time and hence $u(t,x) \le 1$ after a finite time. Let t_0 be the infimum of the values of t for which $\|u(t,x)\|_{L^\infty} \le 1$. If $\overline{u(t_0,x)} < 1$ then for $t \ge t_0$ we are again in the linear situation and hence we have convergence to $u_\infty = \overline{u(t_0,x)}$ and the rate of convergence is exponential. If $\overline{u(t_0,x)} = 1$ then $u(t_0,x) = 1$ a.e. and the limit $u_\infty = 1$ has been reached in finite time. ■

From Proposition 7.7 it follows that if β is forceful then the convergence rate is at least exponential. In order to assure exponential convergence however, β need not be forceful. It is usually sufficient that $\frac{d^+}{ds}\beta|_{s=1} > 0$, this can be seen from the next result.

Proposition 7.8:

Let β satisfy (7.13). If there is a $\gamma > 0$ such that

$$\beta(s) \ge \gamma(s-1) \quad \text{for } s \ge 1 \tag{7.20}$$

then for every $u_0 \in L^2(\Omega)$ the convergence of $u(t,x)$ to u_∞ has exponential rate.

Proof: From Poincaré's inequality and our assumptions we have:

$$(Au, u-Pu) = \int_\Omega |\nabla u|^2 dx + \int_\Omega \beta(u)(u-Pu)dx \ge \gamma_1 \int_\Omega |u-\bar{u}|^2 dx + \gamma \int_{u\ge 1} |u-1||u-Pu|dx \; . \tag{7.21}$$

If $\bar{u}$ satisfies $\bar{u} \le 1$ then by (7.3) $Pu = \bar{u}$ and from (7.21) we have

$$(Au, u-Pu) \ge \gamma_1 \|u - Pu\|^2 \; . \tag{7.22}$$

If $\bar{u} > 1$ then by (7.3) $Pu = 1$ and from (7.21) we have:

$$\begin{aligned}(Au, u-Pu) &\ge \min(\gamma,\gamma_1)\Big(\int_\Omega |u-\bar{u}|^2 dx + \int_{u\ge 1} |u-1|^2 dx\Big) \\ &= \min(\gamma,\gamma_1)\Big(\int_\Omega |u-1|^2 dx - |\Omega||\bar{u}-1|^2 + \int_{u\ge 1}|u-1|^2 dx\Big) \; .\end{aligned} \tag{7.23}$$

Since $\bar{u} > 1$ we have

$$\left(\int_\Omega (u-1)dx\right)^2 \le \left(\int_\Omega (u-1)_+ dx\right)^2$$

where $a_+ = \max(a,0)$. Therefore,

$$|\Omega|^2|\bar{u}-1|^2 = \left(\int_\Omega (u-1)dx\right)^2 \le \left(\int_\Omega (u-1)_+ dx\right)^2 \le |\Omega| \int_{u\ge 1} (u-1)^2 dx \tag{7.24}$$

which together with (7.23) implies

$$(Au, u-Pu) \ge \min(\gamma,\gamma_1) \int_\Omega |u-1|^2 dx = \min(\gamma,\gamma_1)\|u-Pu\|^2 \ . \tag{7.25}$$

Combining (7.22) and (7.25) we have always

$$(Au, u-Pu) \ge \min(\gamma,\gamma_1)\|u-Pu\|^2$$

and the result follows from Proposition 4.7. ∎

We conclude this section with two lower bounds on u_∞.

Proposition 7.9:

Let $u_0(x) \ge 0$ and set $v_0(x) = \min(u_0(x),1)$. If u_∞ is the limit of the solution $u(t,x)$ of (7.1) as $t \to \infty$ then

$$u_\infty \ge \overline{v_0(x)} = \frac{1}{|\Omega|}\int_\Omega v_0(x)dx \ . \tag{7.26}$$

Proof: Consider the comparison function v satisfying

$$\begin{cases} v_t - \Delta v = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \dfrac{\partial v}{\partial n} = 0 & \text{in} \quad \partial\Omega \times (0,\infty) \\ v(0,x) = v_0(x) \ . \end{cases}$$

From the maximum principle it follows that $0 \le v(t,x) \le 1$ a.e. for all $t \ge 0$. From Corollary 7.4 we have $v(t,x) \to \frac{1}{|\Omega|}\int_\Omega v_0(x)dx$ as $t \to \infty$. To conclude the proof we shall now show that $u(t,x) \ge v(t,x)$ for all $t \ge 0$. Let $w = u - v$ then

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + \beta(u) \ni 0 \\ \dfrac{\partial w}{\partial n} = 0 \\ w(0,x) = u_0(x) - v_0(x) \ge 0 \ . \end{cases} \tag{7.27}$$

Multiplying (7.27) by $w_- = -\min(w,0)$ and integrating over Ω we obtain

(7.28) $$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_-|^2 dx + \int_\Omega |\nabla w_-|^2 dx + \int_\Omega \beta(u) w_- dx = 0 .$$

But $w_-(t,x) \neq 0$ only if $u(t,x) \le v(t,x)$. Since $v(t,x) \le 1$, $w_-(t,x) \neq 0$ only where $\beta(u(t,x)) = 0$ and therefore

$$\int_\Omega \beta(u) w_- dx = 0 .$$

So from (7.28) it follows that $\int_\Omega |w_-|^2 dx$ is nonincreasing and since for $t = 0$ it equals zero we have $w_-(t,x) = 0$ a.e. and therefore $u(t,x) \ge v(t,x)$ and the proof if complete. ∎

In certain cases, one can obtain a lower estimate for u_∞ which is different from the estimate given in Proposition 7.9. We restrict ourselves now to the case where $u_0 \in L^\infty(\Omega)$ and $\bar{u}_0 \le 1$ and conclude this section with the following result;

Proposition 7.10:

Let $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$ and denote $m = \|u_0\|_{L^\infty}$. Assume further that

(7.29) $$\beta(s) \le \gamma(s-1)^\rho \quad \text{for} \quad s \ge 1$$

where $\gamma > 0$ and $\rho > 1$. If $\bar{u}_0 \le 1$ then

(7.30) $$u_\infty \ge \bar{u}_0 - \frac{\gamma}{2\gamma_1} \cdot \frac{\delta_0^{2(\rho-1)}}{\rho - 1} (m-1)_+^{2-\rho} \quad \text{for} \quad 1 < \rho \le 2$$

and

(7.31) $$u_\infty \ge \bar{u}_0 - \frac{\gamma}{2\gamma_1} \delta_0^2 (m-1)_+^{\rho-2} \quad \text{for} \quad 2 < \rho$$

where

$$\delta_0^2 = \frac{1}{|\Omega|}\int_\Omega (u_0 - \bar{u}_0)^2 dx$$

is the initial variance of u and γ_1 is the smallest nonzero eigenvalue of $-\Delta$ with the Neumann boundary conditions on $\partial\Omega$.

Proof: Integrating Equation (7.1) over Ω yields

(7.32) $$\frac{d\bar{u}}{dt} + \frac{1}{|\Omega|}\int_\Omega \beta(u) dx = 0 .$$

Using (7.29) we have

$$\frac{d\bar{u}}{dt} + \frac{\gamma}{|\Omega|} \int_{u\geq 1} (u-1)^{\rho} dx \geq 0 . \tag{7.33}$$

From (7.32) it follows that $\bar{u}(t)$ is nonincreasing in t and therefore $\bar{u}(t) \leq \bar{u}_0 \leq 1$. This together with (7.5) implies

$$\int_{u\geq 1} (u-1)^2 dx \leq \int_{u\geq 1} (u-\bar{u})^2 dx \leq \int_{\Omega} (u-\bar{u})^2 dx \leq e^{-2\gamma_1 t} |\Omega| \delta_0^2 . \tag{7.34}$$

Now if $1 < \rho \leq 2$ we have

$$\begin{aligned} \int_{u\geq 1} (u-1)^{\rho} dx &\leq \left(\int_{u\geq 1} (u-1)dx\right)^{2-\rho} \left(\int_{u\geq 1} (u-1)^2 dx\right)^{\rho-1} \leq |\Omega|^{2-\rho} (m-1)_+^{2-\rho} \left(\int_{\Omega} (u-\bar{u})^2 dx\right)^{\rho-1} \\ &\leq |\Omega| (m-1)_+^{2-\rho} e^{-2\gamma_1 (\rho-1) t} \delta_0^{2(\rho-1)} \end{aligned} \tag{7.35}$$

while if $\rho \geq 2$

$$\begin{aligned} \int_{u\geq 1} (u-1)^2 dx &\leq (m-1)_+^{\rho-2} \int_{u\geq 1} (u-1)^2 dx \leq (m-1)_+^{\rho-2} \int_{\Omega} |u-\bar{u}|^2 dx \\ &\leq (m-1)_+^{\rho-2} e^{-2\gamma_1 t} |\Omega| \delta_0^2 . \end{aligned} \tag{7.36}$$

Substituting these estimates into (7.33) and integrating from zero to infinity gives the results. ∎

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The present paper deals with the strong convergence of trajectories $S(t)x$ of a strongly continuous semigroup of contractions $S(t)$, as $t \to \infty$. A general sufficient condition for such convergence to occur is introduced and some examples in which the condition is satisfied are provided. Strengthening the general convergence condition, sufficient conditions for certain rates of convergence of $S(t)x$ to its limit are exhibited. In particular a sufficient condition for a trajectory to reach equilibrium in finite time is given. The convergence as $t \to \infty$ of solutions of certain nonautonomous equations and a discrete version of all the previous results are briefly discussed.

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